

EQUIVARIANT VECTOR BUNDLES AND LOGARITHMIC CONNECTIONS ON TORIC VARIETIES

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ABSTRACT. Let X be a smooth complete complex toric variety such that the boundary is a simple normal crossing divisor, and let E be a holomorphic vector bundle on X . We prove that the following three are equivalent:

- The holomorphic vector bundle E admits an equivariant structure.
- The holomorphic vector bundle E admits an integrable logarithmic connection singular over D .
- The holomorphic vector bundle E admits a logarithmic connection singular over D .

We show that an equivariant vector bundle on X has a tautological integrable logarithmic connection singular over D . This is used in computing the Chern classes of the equivariant vector bundles on X . We also prove a version of the result for holomorphic vector bundles on log parallelizable G -pairs (X, D) , where G is a simply connected complex affine algebraic group.

1. INTRODUCTION

Let X be a smooth complete variety over \mathbb{C} . A reduced effective divisor D on X is called *locally simple normal crossing* if around every point $x \in D$ there are holomorphic coordinate functions $\{z_1, \dots, z_d\}$ on some analytic open subset $U_x \subset X$ containing x such that $x = (0, \dots, 0)$ and

$$D \cap U_x = \{(z_1, \dots, z_d) \mid \prod_{i=1}^{d'} z_i = 0\} \cap U_x$$

for some $d' \leq d$. A locally simple normal crossing divisor is called a *simple normal crossing* divisor if each irreducible component of it is smooth.

Now take X to be a smooth complete complex toric variety. Let \mathcal{T} denote the complex torus action on X defining its toric structure. The dimension of X will be denoted by d . Let $\mathcal{U} \subset X$ be the nonempty Zariski open subset on which the action of \mathcal{T} is free. We impose the following two conditions on X :

- the variety X is smooth, and
- the complement

$$D := X \setminus \mathcal{U}$$

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is a locally simple normal crossing divisor.

From the above assumptions it follows that

- D is a simple normal crossing divisor, and
- the logarithmic tangent bundle $TX(-\log D)$ is holomorphically trivial.

See [Wi, p. 473–474, Main Theorem].

In Section 2 we prove our first main result (see Theorem 2.2):

Theorem 1.1. *Let E be a holomorphic vector bundle over the toric variety X . The following three statements are equivalent:*

- (1) *The holomorphic vector bundle E admits an equivariant structure.*
- (2) *The holomorphic vector bundle E admits an integrable logarithmic connection singular over D .*
- (3) *The holomorphic vector bundle E admits a logarithmic connection singular over D .*

In fact, we show that an equivariant holomorphic vector bundle on X has a tautological integrable logarithmic connection singular over D (see Proposition 2.1).

Using the tautological integrable logarithmic connection on an equivariant holomorphic vector bundle E on the toric variety X , we compute the Chern classes of E (see Section 5). Let N be the lattice associated to the toric variety, and let $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be the dual lattice. The toric variety is defined by a fan Δ . For each one dimensional facet $n \in \Delta_1$, let X_n denote the corresponding boundary divisor. For $m \in M$, we have integers $d_{nm} \geq 0$ defined in (5.6). Then the Chern classes of E are given by

$$c_k(E) = \sum_{\substack{m_1, \dots, m_k \in M \\ m_i \neq m_j}} \prod_{i=1}^k \left(\sum_{n \in \Delta_1} d_{nm_i} \langle m_i, n \rangle [X_n] \right)$$

(see (5.8)).

Theorem 1.1 holds for a larger family of varieties. Let G be a complex affine algebraic group. Let X be a smooth complete complex variety on which G acts. Let $D \subset X$ be a simple normal crossing divisor preserved by the action of G on X . This (X, D) is called a G -pair.

Let \mathfrak{g} be the Lie algebra of G . For a G -pair (X, D) , we have a homomorphism of coherent sheaves

$$\text{op}_{X,D} : X \times \mathfrak{g} \longrightarrow TX(-\log D)$$

induced by the action of G . The G -pair (X, D) is called *log parallelizable* if the above homomorphism $\text{op}_{X,D}$ is an isomorphism. See [Br] for properties of log parallelizable G -pairs.

An equivariant vector bundle on a G -pair (X, D) is a holomorphic vector bundle on X equipped with a lift of the action of G on X .

In Section 3 we prove the following (see Theorem 3.2):

Theorem 1.2. *Assume that G is simply connected. Let (X, D) be a log parallelizable G -pair. Let E be a holomorphic vector bundle on X . The following two statements are equivalent:*

- (1) *The holomorphic vector bundle E admits an equivariant structure.*
- (2) *The holomorphic vector bundle E admits an integrable logarithmic connection singular over D .*

In Section 3.1 we give an example showing that the assumption in Theorem 1.2 that G is simply connected is necessary. A key step in the proof of Theorem 1.1 is a theorem of Klyachko which says that a holomorphic vector bundle E on a toric variety X admits an equivariant structure if and only if all the pullbacks of E by the elements of the torus acting on X are holomorphically isomorphic to E . The example in Section 3.1 also shows that this criterion fails for holomorphic vector bundles on a general log parallelizable G -pair.

We next consider equivariant vector bundles over a complete toric variety X defined over an algebraically closed field of arbitrary characteristic. An equivariant vector bundle E over X is nef (respectively, trivial) if and only if its restriction to any invariant curve $C \subset X$ is nef (respectively, trivial) [HMP]. In Section 4, we see that such an explicit result cannot be expected for *semistable* equivariant bundles. In fact, if an equivariant vector bundle on X is semistable when restricted to any invariant curve, then it is decomposable into a direct sum of copies of a line bundle (Proposition 4.1).

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2. LOGARITHMIC CONNECTION ON EQUIVARIANT BUNDLES

In this section, the base field is taken to be the complex numbers.

Let X to be a smooth complete toric variety of dimension d . Let \mathcal{T} denote the complex torus that acts on X defining its toric structure. Let $\mathcal{U} \subset X$ be the nonempty Zariski open subset on which the action of \mathcal{T} is free. Let

$$D := X \setminus \mathcal{U}$$

be the complement. We assume that D is a simple normal crossing divisor.

Proposition 2.1. *Let E be an equivariant vector bundle on X . Then E has a natural integrable logarithmic connection singular over D .*

Proof. The Lie algebra of the torus \mathcal{T} acting on X will be denoted by \mathfrak{t} . Let

$$V := X \times \mathfrak{t} \longrightarrow X$$

be the trivial vector bundle with fiber \mathfrak{t} . The action of \mathcal{T} on X defines an \mathcal{O}_X -linear homomorphism

$$(2.1) \quad V \longrightarrow TX.$$

We recall that $TX(-\log D)$ is the subsheaf of TX generated by all locally defined holomorphic vector fields that take the subsheaf $\mathcal{O}_X(-D) \subset \mathcal{O}_X$ to $\mathcal{O}_X(-D)$. It is easy to see that the image of the homomorphism in (2.1) is contained in the subsheaf $TX(-\log D) \subset TX$. Indeed, this follows immediately from the fact that the action of \mathcal{T} on X preserves D .

Let

$$(2.2) \quad \beta : V \longrightarrow TX(-\log D)$$

be the homomorphism obtained above from the action of \mathcal{T} on X . We noted earlier that $TX(-\log D)$ is holomorphically trivial by [Wi, p. 473–474, Main Theorem].

Any holomorphic homomorphism of holomorphically trivial vector bundles on X which is isomorphism over some point of X is an isomorphism over X . To see this, for any such homomorphism $\psi : E_1 \longrightarrow E_2$, consider the homomorphism

$$\bigwedge^r \psi : \bigwedge^r E_1 \longrightarrow \bigwedge^r E_2,$$

where r is the rank of E_1 (same as the rank of E_2 because ψ is an isomorphism over some point of X). Therefore, $\bigwedge^r \psi$ defines a nonzero holomorphic section of the holomorphic line bundle $\text{Hom}(\bigwedge^r E_1, \bigwedge^r E_2) = \mathcal{O}_X$. But any holomorphic function on X non identically zero does not vanish anywhere. Hence the homomorphism $\bigwedge^r \psi$ is nowhere vanishing. This implies that ψ is an isomorphism over X .

Since both V and $TX(-\log D)$ are holomorphically trivial, the homomorphism β in (2.2) is an isomorphism.

Let $\text{At}(E)$ be the Atiyah bundle for E . We quickly recall the construction of $\text{At}(E)$. Let

$$(2.3) \quad \delta : E_{\text{GL}} \longrightarrow X$$

be the principal $\text{GL}(r, \mathbb{C})$ -bundle associated to E , where r is the rank of E . Then

$$\text{At}(E) := (\delta_*(TE_{\text{GL}}))^{\text{GL}(r, \mathbb{C})} \subset \delta_*(TE_{\text{GL}})$$

(the action of $\text{GL}(r, \mathbb{C})$ on E_{GL} produces an action of $\text{GL}(r, \mathbb{C})$ on the direct image $\delta_*(TE_{\text{GL}})$). We note that the Lie bracket operation of locally defined holomorphic vector fields on E_{GL} produces a Lie algebra structure on the sheaf of sections of $\text{At}(E)$.

The Atiyah bundle $\text{At}(E)$ fits in a short exact sequence of holomorphic vector bundles over X

$$(2.4) \quad 0 \longrightarrow \text{End}(E) \longrightarrow \text{At}(E) \xrightarrow{\bar{\eta}} TX \longrightarrow 0$$

which is known as the *Atiyah exact sequence* for E . The Atiyah exact sequence produces the short exact sequence

$$(2.5) \quad 0 \longrightarrow \text{End}(E)(-\log D) \longrightarrow \text{At}(E)(-\log D) \xrightarrow{\eta} TX(-\log D) \longrightarrow 0,$$

where

$$(2.6) \quad \text{At}(E)(-\log D) := \bar{\eta}^{-1}(TX(-\log D)).$$

The homomorphism η in (2.5) is the restriction of $\bar{\eta}$. Note that

$$\text{At}(E)(-\log D) = (\delta_*(TE_{\text{GL}}(-\log f^{-1}(D))))^{\text{GL}(r, \mathbb{C})},$$

where δ is the projection in (2.3). Therefore the Lie algebra structure on the sheaf of sections of $\text{At}(E)$ preserves the sheaf of sections of $\text{At}(E)(-\log D)$.

A logarithmic connection on E singular over D is, by definition, a holomorphic splitting of the short exact sequence in (2.5).

The action of \mathcal{T} on E produces an \mathcal{O}_X -linear homomorphism

$$(2.7) \quad V = X \times \mathfrak{t} \longrightarrow \text{At}(E).$$

Since the image of the homomorphism in (2.1) is contained in $TX(-\log D)$, and the diagram

$$\begin{array}{ccc} V & \longrightarrow & \text{At}(E) \\ \parallel & & \downarrow \bar{\eta} \\ V & \longrightarrow & TX \end{array}$$

is commutative (the horizontal maps are as in (2.7) and (2.1), it follows that the homomorphism in (2.7) factors through a homomorphism

$$(2.8) \quad \gamma : V \longrightarrow \text{At}(E)(-\log D) \subset \text{At}(E),$$

where $\text{At}(E)(-\log D)$ is defined in (2.6).

We showed above that the homomorphism β in (2.2) is an isomorphism. Consider the composition

$$\gamma \circ \beta^{-1} : TX(-\log D) \longrightarrow \text{At}(E)(-\log D).$$

Since the action of \mathcal{T} on E is a lift of the action of \mathcal{T} on X , it follows that

$$\eta \circ (\gamma \circ \beta^{-1}) = \text{Id}_{TX(-\log D)},$$

where η is the homomorphism in (2.5). Therefore, $\gamma \circ \beta^{-1}$ is a logarithmic connection on E singular over D .

The homomorphism in (2.1) takes the Lie algebra structure on the fibers of V (recall that the fibers of V are the abelian Lie algebra \mathfrak{t}) to the Lie algebra structure on the sheaf of holomorphic vector fields on X defined by the Lie bracket operation. Similarly, the homomorphism in (2.7) takes the Lie algebra structure on the fibers of V to the Lie algebra structure on the sheaf of sections of $\text{At}(E)$ (this Lie algebra structure was explained earlier). Consequently, the homomorphism $\gamma \circ \beta^{-1}$ is compatible with the Lie algebra structure on the sheaf of sections of $TX(-\log D)$ and $\text{At}(E)(-\log D)$ (recall that

the Lie algebra structure of the sheaf of sections of $\text{At}(E)$ preserves the sheaf of sections of $\text{At}(E)(-\log D)$. Therefore, the logarithmic connection on E defined by $\gamma \circ \beta^{-1}$ is integrable. \square

Theorem 2.2. *Let E be a holomorphic vector bundle over the toric variety X . The following three statements are equivalent:*

- (1) *The vector bundle E admits an equivariant structure.*
- (2) *The vector bundle E admits an integrable logarithmic connection singular over D .*
- (3) *The vector bundle E admits a logarithmic connection singular over D .*

Proof. From Proposition 2.1 we know that the first statement implies the second statement. The second statement immediately implies the third statement. So it is enough to prove that the third statement implies the first statement.

Let

$$\rho : \mathcal{T} \longrightarrow \text{Aut}(X)$$

be the action of \mathcal{T} on X .

Assume that E admits a logarithmic connection singular over D .

It is known that a holomorphic vector bundle W on X admits an equivariant structure if for every $g \in \mathcal{T}$, the pulled back vector bundle $\rho(g)^*W$ is holomorphically isomorphic to W [K1, p. 342, Proposition 1.2.1].

We will show that the pulled back vector bundle $\rho(g)^*E$ is holomorphically isomorphic to E for all $g \in \mathcal{T}$.

Let \mathcal{G}_E be the set of all pairs of the form (f, ϕ) , where $f \in \text{Aut}(X)$ and ϕ is a holomorphic automorphism of the vector bundle E over the automorphism f . So f and ϕ fit in a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & X \end{array}$$

such that ϕ is fiberwise linear. Note that ϕ defines a holomorphic isomorphism of E with the pullback f^*E . The natural composition operation makes \mathcal{G}_E a group. The inverse of an element (f, ϕ) is (f^{-1}, ϕ^{-1}) . This \mathcal{G}_E is a complex Lie group with Lie algebra $H^0(X, \text{At}(E))$. We recall that the Lie algebra structure of $H^0(X, \text{At}(E))$ is given by the Lie bracket of holomorphic vector fields.

Let

$$(2.9) \quad p : \mathcal{G}_E \longrightarrow \text{Aut}(X)$$

be the homomorphism defined by $(f, \phi) \longmapsto f$. The corresponding homomorphism of Lie algebras

$$(2.10) \quad dp : \text{Lie}(\mathcal{G}_E) = H^0(X, \text{At}(E)) \longrightarrow \text{Lie}(\text{Aut}(X)) = H^0(X, TX)$$

coincides with the homomorphism $H^0(X, \text{At}(E)) \longrightarrow H^0(X, TX)$ induced by $\bar{\eta}$ in (2.4).

Consider $\text{At}(E)(-\log D)$ defined in (2.6). Let

$$\nabla : TX(-\log D) \longrightarrow \text{At}(E)(-\log D)$$

be a logarithmic connection on E singular over D . Let

$$\nabla' : H^0(X, TX(-\log D)) \longrightarrow H^0(X, \text{At}(E)(-\log D))$$

be the homomorphism induced by ∇ . We recall that $H^0(X, TX(-\log D))$ is identified with the Lie algebra \mathfrak{t} using the action of \mathcal{T} on X (the isomorphism β in (2.2) produces the identification). Since $\eta \circ \nabla = \text{Id}_{TX(-\log D)}$, where η is the homomorphism in (2.5), it follows that the composition with dp in (2.10)

$$(dp) \circ \nabla' : H^0(X, TX(-\log D)) \longrightarrow H^0(X, TX)$$

actually coincides with the homomorphism given by the inclusion $TX(-\log D) \hookrightarrow TX$ (recall that $\text{At}(E)(-\log D)$ is a subsheaf of $\text{At}(E)$).

Consequently, the image of dp contains $H^0(X, TX(-\log D))$. Since the group \mathcal{T} is connected, and the homomorphism β in (2.2) is an isomorphism, this implies that the image of the homomorphism p in (2.9) contains $\rho(\mathcal{T})$. Therefore, for each $g \in \mathcal{T}$, there is a holomorphic isomorphism between $\rho(g)^*E$ and E . Now by the earlier mentioned criterion of Klyachko, the holomorphic vector bundle E admits an equivariant structure. This completes the proof. \square

2.1. Equivariant Krull–Schmidt decomposition. An equivariant vector bundle on a smooth complete complex toric variety X is called *decomposable* if it is a direct sum of two equivariant vector bundles of positive rank. An equivariant vector bundle is called *indecomposable* if it is not decomposable.

For an equivariant vector bundle E on X , let $\text{Aut}^{\mathcal{T}}(E)$ denote the group of all holomorphic equivariant automorphisms of the vector bundle E . We note that E is indecomposable if and only if the maximal torus of $\text{Aut}^{\mathcal{T}}(E)$ is of dimension one (in other words, isomorphic to \mathbb{C}^*).

Corollary 2.3. *Let E be an equivariant vector bundle over a smooth complete complex toric variety X . Let*

$$E = \bigoplus_{i=1}^m E_i \quad \text{and} \quad E = \bigoplus_{i=1}^n F_i$$

be two decompositions of E into direct sum of indecomposable equivariant vector bundles. Then $m = n$, and there is a permutation σ of $\{1, \dots, m\}$ such that the equivariant vector bundle E_i is isomorphic to the equivariant vector bundle $F_{\sigma(i)}$ for every $i \in \{1, \dots, m\}$.

This follows from Theorem 4.1 of [BP].

3. EQUIVARIANT BUNDLES ON G -PAIRS

Let G be a connected complex affine algebraic group.

Let X be a smooth complete complex variety equipped with an action of G . Let $D \subset X$ be a simple normal crossing divisor preserved by the action of G on X . This (X, D) is called a G -pair.

The Lie algebra of G will be denoted by \mathfrak{g} . For a G -pair (X, D) , we have a homomorphism of coherent sheaves

$$(3.1) \quad \text{op}_{X,D} : X \times \mathfrak{g} \longrightarrow TX(-\log D)$$

induced by the action of G , where $X \times \mathfrak{g}$ is the trivial holomorphic vector bundle on X with fiber \mathfrak{g} . The G -pair (X, D) is called *log parallelizable* if the above homomorphism $\text{op}_{X,D}$ is an isomorphism.

An equivariant vector bundle on a G -pair (X, D) is a holomorphic vector bundle on X equipped with a lift of the action of G on X .

Proposition 3.1. *Let (X, D) be a log parallelizable G -pair. Let E be a G -equivariant vector bundle on X . Then E has a natural integrable logarithmic connection singular over D .*

Proof. The proof is identical to the proof of Proposition 2.1. The isomorphism β used in the proof of Proposition 2.1 is now replaced by the isomorphism $\text{op}_{X,D}$ in (3.1). Note that the structure of the Lie algebra \mathfrak{t} does not play any rôle in the proof of Proposition 2.1. \square

The following result is an analog of Theorem 2.2.

Theorem 3.2. *Assume that the group G is simply connected. Let (X, D) be a log parallelizable G -pair. Let E be a holomorphic vector bundle on X . The following two statements are equivalent:*

- (1) *The vector bundle E admits an equivariant structure.*
- (2) *The vector bundle E admits an integrable logarithmic connection singular over D .*

Proof. The second statement follows from the first statement by Proposition 3.1.

To prove the converse, assume that the vector bundle E admits an integrable logarithmic connection singular over D .

As in the proof of Theorem 2.2, let \mathcal{G}_E be the complex group consisting of all pairs of the form (f, ϕ) , where $f \in \text{Aut}(X)$ and ϕ is a holomorphic automorphism of the vector bundle E over f . Consider

$$dp : \text{Lie}(\mathcal{G}_E) = H^0(X, \text{At}(E)) \longrightarrow \text{Lie}(\text{Aut}(X)) = H^0(X, TX)$$

constructed in (2.10). Let $\nabla : TX(-\log D) \rightarrow \text{At}(E)(-\log D)$ be a logarithmic connection on E singular over D . Let

$$(3.2) \quad \nabla' : H^0(X, TX(-\log D)) \rightarrow H^0(X, \text{At}(E)(-\log D))$$

be the homomorphism induced by ∇ . As ∇ is an integrable connection, the map in (3.2) is a homomorphism of Lie algebras.

Using the isomorphism $\text{op}_{X,D}$ in (3.1), the Lie algebra \mathfrak{g} of G is identified with the Lie algebra $H^0(X, TX(-\log D))$. Therefore, we get a homomorphism of Lie algebras

$$\mathfrak{g} = H^0(X, TX(-\log D)) \xrightarrow{\nabla'} H^0(X, \text{At}(E)(-\log D)) \hookrightarrow H^0(X, \text{At}(E)) = \text{Lie}(\mathcal{G}_E).$$

Since G is simply connected, this homomorphism of Lie algebras $\mathfrak{g} \rightarrow \text{Lie}(\mathcal{G}_E)$ integrates into a homomorphism of Lie groups

$$\beta : G \rightarrow \mathcal{G}_E.$$

This homomorphism β evidently defines an equivariant structure on E . \square

The condition in Theorem 3.2 that G is simply connected is necessary; see Section 3.1 below.

The proof that the third statement in Theorem 2.2 implies the first statement in Theorem 2.2 breaks down for log parallelizable G -pair due to the following reason: Proposition 1.2.1 of [Kl], which is crucially used in the proof of Theorem 2.2, is not valid for a general log parallelizable G -pair; see Section 3.1 below for such an example.

3.1. An example. Take $G = \text{PGL}(2, \mathbb{C})$. Let $\mathbb{P}(M(2, \mathbb{C}))$ be the projective space parametrizing lines in the 2×2 complex matrices. Let

$$f : \text{PGL}(2, \mathbb{C}) \rightarrow \mathbb{P}(M(2, \mathbb{C})) = \mathbb{CP}^3$$

be the natural embedding defined by the inclusion map of $\text{GL}(2, \mathbb{C})$ in the vector space $M(2, \mathbb{C})$. The right and left translation actions of $\text{GL}(2, \mathbb{C})$ on $M(2, \mathbb{C})$ produce respectively right and left actions of $\text{PGL}(2, \mathbb{C})$ on $\mathbb{P}(M(2, \mathbb{C}))$. The above embedding f is the wonderful compactification of $\text{PGL}(2, \mathbb{C})$ (see [DP1], [DP2]), but we do not use this.

Let

$$D := \mathbb{P}(M(2, \mathbb{C})) \setminus f(\text{PGL}(2, \mathbb{C})) \subset \mathbb{P}(M(2, \mathbb{C})) := X$$

be the boundary divisor. We will show that the vector bundle $TX(-\log D)$ is holomorphically trivial.

Note that D is a smooth hypersurface of degree two. Let $\iota : D \hookrightarrow X$ be the inclusion map. Consider the short exact sequence of coherent sheaves on X

$$0 \rightarrow TX(-\log D) \rightarrow TX \rightarrow \iota_* N_D = \mathcal{O}_X(D)|_D \rightarrow 0,$$

where N_D is the normal bundle of D ; the above isomorphism

$$\iota_* N_D = \mathcal{O}_X(D)|_D$$

is the Poincaré adjunction formula. Since the degree of D is two, we have

$$\text{degree}(\mathcal{O}_X(D)|_D) = 4.$$

Hence from the above short exact sequence it follows that

$$\text{degree}(TX(-\log D)) = 0.$$

Let $V = X \times \mathfrak{sl}(2, \mathbb{C})$ be the trivial holomorphic vector bundle over X with fiber $\mathfrak{sl}(2, \mathbb{C}) = \text{Lie}(\text{PGL}(2, \mathbb{C}))$. The action of $\text{PGL}(2, \mathbb{C})$ on X defines a homomorphism

$$\beta : V \longrightarrow TX(-\log D).$$

Let

$$\bigwedge^2 \beta : \bigwedge^2 V = \mathcal{O}_X \longrightarrow \bigwedge^2 (TX(-\log D)) = \mathcal{O}_X$$

be the homomorphism induced by β . Since the homomorphism $\bigwedge^2 \beta$ is an isomorphism over the complement of D , it follows that $\bigwedge^2 \beta$ is an isomorphism over X (a somewhere nonzero holomorphic function on X is nowhere vanishing). Therefore, β is an isomorphism over X .

Consequently, (X, D) is a log parallelizable $\text{PGL}(2, \mathbb{C})$ -pair.

Let $L := \mathcal{O}_{\mathbb{P}(M(2, \mathbb{C}))}(-1)$ be the tautological line bundle on X . We will show that L admits an integrable logarithmic connection singular over D . For this, note that the line bundle $L^{\otimes 2} = \mathcal{O}_X(-D)$ has a tautological integrable logarithmic connection, singular over D , given by the de Rham differential $\alpha \mapsto d\alpha$. For any holomorphic line bundle ξ , and any nonzero integer m , a logarithmic connection on ξ induces a logarithmic connection on $\xi^{\otimes m}$; moreover, this map is a bijection between the logarithmic connections on ξ and logarithmic connections on $\xi^{\otimes m}$. Therefore, the above logarithmic connection on $L^{\otimes 2}$ induces an integrable logarithmic connection on L singular over D .

As before, let \mathcal{G}_L be the complex group consisting of all pairs of the form (h, ϕ) , where $h \in \text{Aut}(X)$ and ϕ is a holomorphic automorphism of the line bundle L over the automorphism h of X . It is easy to see that \mathcal{G}_L is identified with $\text{GL}(2, \mathbb{C})$. Indeed, the tautological action of $\text{GL}(2, \mathbb{C})$ on $\mathcal{O}_{\mathbb{P}(M(2, \mathbb{C}))}(-1)$ produces this isomorphism.

The holomorphic line bundle L does not admit an equivariant structure. This follows immediately from the fact that there is no nontrivial homomorphism from $\text{PGL}(2, \mathbb{C})$ to $\text{GL}(2, \mathbb{C})$ (in particular, the projection homomorphism

$$\text{GL}(2, \mathbb{C}) \longrightarrow \text{PGL}(2, \mathbb{C})$$

does not split). Therefore, the condition in Theorem 3.2 that G is simply connected is essential.

The pullback of $L = \mathcal{O}_{\mathbb{P}(M(2, \mathbb{C}))}(-1)$ by any automorphism of $\mathbb{P}(M(2, \mathbb{C}))$ is holomorphically isomorphic to L . But, as we have seen above, the holomorphic line bundle L does not admit an equivariant structure. Therefore, Proposition 1.2.1 of [Kl] is not valid for a general log parallelizable G -pair.

4. SEMISTABILITY AND RESTRICTION TO INVARIANT CURVES

Let X be a complete toric variety defined over an algebraically closed field k . We do not make any assumption on the characteristic of k . By an invariant curve in X we will mean pair (C, f) , where C is an irreducible smooth projective curve defined over k , and $f : C \rightarrow X$ is a separable morphism, which is an embedding over a nonempty open subset of C , such that $f(C)$ is preserved by the action of the torus on X . If (C, f) is an invariant curve, then C is a rational curve.

Proposition 4.1. *Let E be an equivariant vector bundle of rank r over X . The following two statements are equivalent:*

- (1) *For every invariant curve (C, f) , the pullback f^*E is semistable.*
- (2) *There is a line bundle L over X such that the vector bundle E is isomorphic to $L^{\oplus r}$.*

Proof. The second statement in the proposition evidently implies the first statement.

Assume that the first statement in the proposition holds. Let $\text{End}(E) = E \otimes E^* \rightarrow X$ be the endomorphism bundle. Take any invariant curve (C, f) . Since C is isomorphic to \mathbb{P}_k^1 , the vector bundle f^*E splits into a direct sum of line bundles [Gr]. Therefore, the given condition that f^*E is semistable implies that f^*E is isomorphic to $\xi^{\oplus r}$ for some line bundle ξ on C . Hence the vector bundle $f^*\text{End}(E) = \text{End}(f^*E)$ is trivial. From this it follows that the vector bundle $\text{End}(E)$ is trivial [HMP, p. 633, Theorem 6.4].

All functions on X are constants. So for any global endomorphism ϕ of E , the coefficients of the characteristic polynomial of ϕ are constant functions. Hence the set of eigenvalues of $\phi_x \in \text{End}(E_x)$ is independent of $x \in X$.

Let

$$\phi \in H^0(X, \text{End}(E)) = k^{r^2}$$

be a section such that the eigenvalues of ϕ are distinct.

The eigenspace decomposition of ϕ_x , $x \in X$, produces a decomposition

$$E = \bigoplus_{i=1}^r L_i$$

into a direct sum of line bundles. From this it follows that

$$(4.1) \quad \text{End}(E) = \bigoplus_{i,j=1}^r L_j \otimes L_i^*.$$

On the other hand,

$$(4.2) \quad \text{End}(E) = \mathcal{O}_X^{\oplus r^2}.$$

Comparing (4.1) and (4.2), from [At, p. 315, Theorem 2] we conclude that $L_j \otimes L_i^* = \mathcal{O}_X$ for all i and j . In other words, $L_i \cong L_j$ for all $1 \leq i, j \leq r$. Therefore, the second statement in the proposition holds. \square

Remark 4.2. The isomorphism in statement (2) in Proposition 4.1 is not necessarily equivariant. To see this, take a line bundle $L \rightarrow X$ with two (non-isomorphic) different toric actions; for instance, take $X = \mathbb{P}^1$, $L = \mathbb{P}^1 \times \mathbb{C}$, $\varphi_1((x, y), t) = (tx, y)$ and $\varphi_2((x, y), t) = (tx, ty)$. Denote L_1, L_2 the two corresponding equivariant line bundles. Then $E = L_1 \oplus L_2$ satisfies the statements in Proposition 4.1. However, there is no equivariant line bundle L' such that $E \cong L' \oplus L'$. Compare with [Kl, Corollary 1.2.4].

5. CHERN CLASSES OF EQUIVARIANT BUNDLES

Let X be a smooth complete complex toric variety of dimension d so that the boundary is a simple normal crossing divisor. In [Kl, Theorem 3.2.1], the Chern classes of an equivariant vector bundle $E \rightarrow X$ are computed through a resolution. In this section we will compute the Chern classes of an equivariant vector bundle E through the natural logarithmic connection on E . As the Newton classes $N_p(E)$ (the sum over the p -th powers of the Chern roots of E) can be expressed in terms of the residues of an integrable logarithmic connection (see [EV, Corollary B.3]), we use such formula to compute the Chern character in our case.

We begin with introducing some notation. The fan associated to X in $N \cong \mathbb{Z}^d$ will be denoted by Δ . The complex torus acting on X is $\mathcal{T} = T_N = N \otimes \mathbb{C}^*$. As before, let $\mathcal{U} \subset X$ be the dense open orbit of \mathcal{T} , and denote

$$D = X \setminus \mathcal{U},$$

which is a normal crossing divisor, by assumption.

For any cone $\sigma \in \Delta$, we have an open subset

$$(5.1) \quad \mathcal{U}_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^*)^{d-k},$$

where $k = \dim \sigma$. This open subset has a distinguished point $x_\sigma \in \mathcal{U}_\sigma$ and its coordinates under the identification (5.1) are $x_\sigma = (0, \dots, 0, 1, \overset{(k)}{\dots}, \overset{(d-k)}{\dots}, 1)$. The orbit of x_σ is denoted by $\mathcal{O}_\sigma = \mathcal{T} \cdot x_\sigma$. This is a torus of dimension $d - k$. The stabilizer of such a point is the subgroup

$$T_\sigma = \ker(T_N \rightarrow \mathcal{O}_\sigma) \subset T_N.$$

The closure of \mathcal{O}_σ in X will be denoted by X_σ ; it is also a toric variety. For one dimensional cone $\delta \in \Delta_1$, such subvarieties X_δ correspond to divisors, and

$$D = \bigcup_{\delta \in \Delta_1} X_\delta.$$

Let E be an equivariant vector bundle of rank r on X , and let ∇ be the integrable logarithmic connection on E constructed in Proposition 2.1. For any cone $\sigma \in \Delta$, the equivariant structure is given locally by a representation $\rho_\sigma : T_\sigma \rightarrow \mathrm{GL}(V)$ which

extends to T_N . Remember that the linear representation ρ_σ splits into isotypical components

$$(5.2) \quad V = \bigoplus_{\chi \in M} V^\sigma(\chi),$$

where $\rho_\sigma(t)v = \chi(t) \cdot v$ for $v \in V^\sigma(\chi)$. Here $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ is the dual lattice of N .

Fix a 1-dimensional cone $\delta \in \Delta$ generated by some $n \in N$. As a result of (5.2) there is a basis $v_1, \dots, v_r \in V$ where $v_i \in V(\chi_i)$ and $\chi_i = \chi_{m_i}$, $m_i \in M$, $i = 1, \dots, r$. The sections $v_i(t) = \rho_\delta(t) \cdot v_i \in \Gamma(\mathcal{U}, E)$ satisfy the condition $\nabla v_i = 0$ and extend to $\mathcal{U}_\delta = \mathcal{U} \cup X_\delta$. These extended sections are denoted by v_i . The section $s_i(t) = \chi_i(t)^{-1} v_i(t) \in \Gamma(\mathcal{U} \cup X_\delta, E)$ is a non-vanishing one and

$$(5.3) \quad \nabla s_i = \frac{d\chi_i^{-1}}{\chi_i^{-1}} s_i = -\frac{d\chi_{m_i}}{\chi_{m_i}} s_i.$$

gives the following local expression for ∇

$$(5.4) \quad \omega_\delta = -\text{diag} \left(\frac{d\chi_{m_i}}{\chi_{m_i}}; i = 1, \dots, r \right).$$

Let $\text{Res}_\delta : E(-\log D) \rightarrow E|_{X_\delta}$ be the Poincaré residue map, and let

$$\Gamma_\delta = \text{Res}_\delta \circ \nabla$$

be the function defined on [EV, Appendix B]. Clearly $\Gamma_\delta(s_i) = -\langle m_i, n \rangle (s_i|_{X_\delta})$.

Let $\Delta_1 = \{\delta_1, \dots, \delta_s\}$ be the set of one-dimensional facets where the i -th facet is generated by a primitive element $n_i \in N$. Recall the isotypical decomposition (5.2) associated to ρ_{δ_j}

$$V = \bigoplus_{i=1}^r V^j(\chi_{m_{i,j}}).$$

Then,

$$(5.5) \quad \Gamma_{\delta_1}^{\alpha_1} \circ \dots \circ \Gamma_{\delta_s}^{\alpha_s} = (-1)^{\alpha_1 + \dots + \alpha_s} \text{diag} \left(\prod_{j=1}^s \langle m_{i,j}, n_j \rangle^{\alpha_j} : i = 1, \dots, r \right),$$

for $\alpha_j \geq 0$, $j = 1, \dots, s$. Now we apply the formula in [EV, Cor. B.3],

$$N_p(E) = (-1)^p \sum_{\alpha_1 + \dots + \alpha_s = p} \frac{p!}{\alpha_1! \dots \alpha_s!} \text{Tr} (\Gamma_{\delta_1}^{\alpha_1} \circ \dots \circ \Gamma_{\delta_s}^{\alpha_s}) [X_{\delta_1}]^{\alpha_1} \dots [X_{\delta_s}]^{\alpha_s},$$

to compute the Newton classes $N_p(E)$ of E , i.e, the sum of p -th powers of the Chern roots.

Using (5.5) we have

$$\begin{aligned}
N_p(E) &= \sum_{\alpha_1 + \dots + \alpha_s = p} \frac{p!}{\alpha_1! \dots \alpha_s!} \left(\sum_{i=1}^r \left(\prod_{j=1}^s \langle m_{i,j}, n_j \rangle^{\alpha_j} \right) [X_{\delta_1}]^{\alpha_1} \dots [X_{\delta_s}]^{\alpha_s} \right) \\
&= \sum_{i=1}^r \sum_{\alpha_1 + \dots + \alpha_s = p} \frac{p!}{\alpha_1! \dots \alpha_s!} ((\langle m_{i,1}, n_1 \rangle [X_{\delta_1}])^{\alpha_1} \dots (\langle m_{i,s}, n_s \rangle [X_{\delta_s}])^{\alpha_s}) \\
&= \sum_{i=1}^r \left(\sum_{j=1}^s \langle m_{i,j}, n_j \rangle [X_j] \right)^p.
\end{aligned}$$

Set

$$(5.6) \quad d_{nm} = \dim V^\delta(\chi_m),$$

where $\delta = \mathbb{R}_+ \cdot n$. Then we enlarge the rank of summation as follows:

$$(5.7) \quad N_p(E) = \sum_{m \in M} \left(\sum_{n \in \Delta_1} d_{nm} \langle m, n \rangle [X_n] \right)^p.$$

where X_n is the divisor associated to the one-dimensional facet $\delta = \mathbb{R}_+ \cdot n$.

This gives the Chern character

$$\text{ch}(E) = \sum_i \exp(\xi_i) = \sum_{p=0}^{\infty} \frac{1}{p!} N_p(E),$$

where ξ_i are the Chern roots. Here

$$\begin{aligned}
\text{ch}(E) &= \sum_{p=0}^{\infty} \sum_{m \in M} \frac{1}{p!} \left(\sum_{n \in \Delta_1} d_{nm} \langle m, n \rangle [X_n] \right)^p \\
&= \sum_{m \in M} \exp \left(\sum_{n \in \Delta_1} d_{nm} \langle m, n \rangle [X_n] \right) \\
&= \sum_{m \in M} \prod_{n \in \Delta_1} \exp(d_{nm} \langle m, n \rangle [X_n]).
\end{aligned}$$

From (5.7), one easily compute the Chern classes. For $m \in M$, define

$$L_m := \sum_{n \in \Delta_1} d_{nm} \langle m, n \rangle [X_n].$$

Thus, $N_p(E) = \sum_{m \in M} L_m^p$ is the sum of p -powers over M which are by definition the Newton polynomials (note that this is a finite sum). Hence L_m are the Chern roots. Now it is straightforward to check that

$$(5.8) \quad c_k(E) = \sum_{\substack{m_1, \dots, m_k \in M \\ m_i \neq m_j}} \prod_{i=1}^k L_{m_i} = \sum_{\substack{m_1, \dots, m_k \in M \\ m_i \neq m_j}} \prod_{i=1}^k \left(\sum_{n \in \Delta_1} d_{nm_i} \langle m_i, n \rangle [X_n] \right).$$

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